# Spinor Transforms in QFT

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Page numbers and equation numbers with dashes in them below references those in Klauber, *Student Friendly QFT Volume 1, Basic Principles and QED,* and *Volume 2, The Standard Model*.

## 1 Background

### 1.1 Weyl Representation Relations

In the Weyl rep, from Vol. 2, pg. 138,

$$ {}_W\psi = \begin{bmatrix} {}_W\psi^L \\ {}_W\psi^R \end{bmatrix} = \begin{bmatrix} {}_W\psi_1 \\ {}_W\psi_2 \\ {}_W\psi_3 \\ {}_W\psi_4 \end{bmatrix}, \tag{1} $$

where the Weyl spinor is composed of a two-component left chiral (L superscript) spinor (also called a *L* Weyl spinor) and a two-component right chiral (R superscript) spinor (also called a *R* Weyl spinor).

Weyl rep gamma matrices, from vol. 2, pgs. 136 and 160, are

$$ {}_W\gamma^0 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} \quad {}_W\gamma^1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \quad {}_W\gamma^2 = \begin{bmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{bmatrix} \quad {}_W\gamma^3 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix} $$

$$ {}_W\gamma^5 = \begin{bmatrix} -I & \\ & I \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{(5-10) and (5-88)} $$

### 1.2 Lorentz Transformation Group for Spinors

A general Lorentz transformation of spinors, including 3D rotations and boosts, is effected by the operator *D,* as shown in Vol. 2, pg. 153, LHS of (5-64) (and Vol. 1, pg. 171, (6-21))

$$ \psi' = D\psi \ , \tag{2} $$

where, in Vol. 2, pg. 153, RHS of (5-64) (or Vol. 1 pg. 171, eq (6-22)), we stated without proof,

$$ D = e^{-i\left(L^k\Theta^k + M^k Q^k\right)} \tag{5-64} $$

$$ L^k = -\tfrac{i}{2}\varepsilon_{ij}{}^k\gamma^i\gamma^j\,,\ \ \Theta^k = \underbrace{\left(\theta^1,\theta^2,\theta^3\right)}_{\substack{\text{rotation}\\ \text{angles}}},\ \ M^k = -\tfrac{i}{2}\gamma^0\gamma^k\,,\ \ Q^k = \underbrace{\underbrace{\left(v^1,v^2,v^3\right)}_{\text{boost velocity comps}}}_{\text{for } v \ll 1} \ . \tag{5-65} $$

Note that (5-65) is for low velocities (and this is a correction listed in the corrections lists). For boost velocity a substantial fraction of the speed of light, $Q^k$ takes a more complicated form. See the appendix herein.

## 2 Proofs in References

Relations (5-64) and (5-65) above are proven in the references of the footnote on the cited page of Vol. 1 (and also in S. Coleman, *Quantum Field Theory, Lectures of Sidney Coleman*, World Scientific 2019, pgs. 369-377), but they done first for the LC spinor field in its own *SU*(2) space, then for the RC spinor fields in its separate *SU*(2) space. As can be seen by comparing the following with those derivations, (5-64) and (5-65) above express the same results in 4D spinor space instead of two separate 2D spaces.

## 3 3D Rotations

Dropping the *w* subscripts on the gamma matrices, but remembering we are working in the Weyl rep, we have, for a rotation only, without boost, from (5-64).

$$ D = e^{-iL^k\Theta^k} \ , \tag{3} $$

where, from the LHS of (5-65),

$$L^1 = -\tfrac{i}{2}\left(\gamma^2\gamma^3 - \gamma^3\gamma^2\right) \qquad L^2 = -\tfrac{i}{2}\left(\gamma^3\gamma^1 - \gamma^1\gamma^3\right) \qquad L^3 = -\tfrac{i}{2}\left(\gamma^1\gamma^2 - \gamma^1\gamma^2\right). \tag{4}$$

From Vol. 1, pg. 414, where the commutation relations for gamma matrices are shown, we have

$$L^3 = -\tfrac{i}{2}\left(\gamma^1\gamma^2 - \gamma^1\gamma^2\right) = -\begin{bmatrix}\sigma_3 & \\ & \sigma_3\end{bmatrix} = -\sigma_3\begin{bmatrix}1 & \\ & 1\end{bmatrix} \qquad L^1 = -\begin{bmatrix}\sigma_1 & \\ & \sigma_1\end{bmatrix} = -\sigma_1\begin{bmatrix}1 & \\ & 1\end{bmatrix} \qquad L^2 = -\begin{bmatrix}\sigma_2 & \\ & \sigma_2\end{bmatrix} = -\sigma_2\begin{bmatrix}1 & \\ & 1\end{bmatrix}, \tag{5}$$

or, generally,

$$L^k = -\sigma_k\begin{bmatrix}1 & \\ & 1\end{bmatrix}. \tag{6}$$

So, (3) becomes

$$D = e^{-iL^k\Theta^k} = e^{i\sigma_k\begin{bmatrix}1 & \\ & 1\end{bmatrix}\theta^k} \tag{7}$$

For small rotations

$$D\psi = e^{i\sigma_k\begin{bmatrix}1 & \\ & 1\end{bmatrix}\theta^k}\begin{bmatrix}\psi^L \\ \psi^R\end{bmatrix} \approx \left(I + i\sigma_k\begin{bmatrix}1 & \\ & 1\end{bmatrix}\theta^k\right)\begin{bmatrix}\psi^L \\ \psi^R\end{bmatrix} \tag{8}$$

and the effect of a rotation on the R Weyl spinor is the same as that on the L Weyl spinor.

As noted, this effect is typically derived in texts separately on each of the two $SU(2)$ spinors $\psi^L$ and $\psi^R$, each in its own $SU(2)$ space. Here, we have shown those relations in the full 4D spinor space of QFT.

## 4 Lorentz Boosts

From (5-64) with boost, but no rotation,

$$D = e^{-\frac{1}{2}\gamma^0\gamma^k Q^k}. \tag{9}$$

where

$$\begin{aligned}
\tfrac{1}{2}\gamma^0\gamma^1 &= \tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} & -1 & & \\ -1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}-\sigma_1 & \\ & \sigma_1\end{bmatrix} \\
\tfrac{1}{2}\gamma^0\gamma^2 &= \tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} & i & & \\ -i & & & \\ & & & -i \\ & & i & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}-\sigma_2 & \\ & \sigma_2\end{bmatrix} \\
\tfrac{1}{2}\gamma^0\gamma^3 &= \tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}-1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}-\sigma_3 & \\ & \sigma_3\end{bmatrix}.
\end{aligned} \tag{10}$$

Or, generally,

$$\tfrac{1}{2}\gamma^0\gamma^k = \tfrac{1}{2}\begin{bmatrix}-\sigma_k & \\ & \sigma_k\end{bmatrix} = -\tfrac{1}{2}\sigma_k\begin{bmatrix}1 & \\ & -1\end{bmatrix}. \tag{11}$$

For a small boost, from (5-65), (9), and (11),

$$D = e^{-\frac{1}{2}\gamma^0\gamma^k v^k} = e^{\frac{1}{2}\sigma_k\begin{bmatrix}1 & \\ & -1\end{bmatrix}v^k} \approx I + \tfrac{1}{2}\sigma_k\begin{bmatrix}1 & \\ & -1\end{bmatrix}v^k \qquad v^k \ll 1. \tag{12}$$

so, from (1), (2), and (12),

$$D\psi = e^{-\frac{1}{2}\gamma^0\gamma^k v^k}\begin{bmatrix}\psi^L \\ \psi^R\end{bmatrix} \approx \left(I + \tfrac{1}{2}\sigma_k\begin{bmatrix}1 & \\ & -1\end{bmatrix}v^k\right)\begin{bmatrix}\psi^L \\ \psi^R\end{bmatrix} \qquad v^k \ll 1, \tag{13}$$

and the effect of a boost on the R Weyl spinor is the opposite of that on the L Weyl spinor.

As is the case for rotation, this effect is typically derived in texts separately on each of the two $SU(2)$ spinors $\psi^L$ and $\psi^R$, each in its own $SU(2)$ space. Here, we have shown those relations in the full 4D spinor space of QFT.

## 5 Visualizing the Transformations

In this section, we show a heuristic visualization of a particle undergoing first a rotation and then, a boost. But, we should first note one thing about spins and momentum of RC (right chiral) and LC (left chiral) fermions.

### 5.1 Spin and Momentum of RC and LC Particles

From Vol. 2, pg. 139, (5-20), we know that the spin operator in the Weyl rep has form

$$\Sigma_3 = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}. \tag{14}$$

So, the action of (14) on (1) is the same on the top two components of $\psi$, as on the bottom two components.

$$\Sigma_3 \psi = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}. \tag{15}$$

At speeds approaching the speed of light, chirality is the same as helicity. Consider, in that case, RC and LC particles to each have spin aligned with momentum **p**. From (15), they will have the same spin. But, at the speed of light, RC and LC particles are also RH (right hand helicity) and LH (left hand helicity) particles, so if they have the same spin, they must have opposite direction momentum.

We can generalize to any speed. That is, RC and LC particles with spin in the same direction have momentum in the opposite directions. We show this in Fig. 1 of the following section.

### 5.2 3D Rotation Visualization

In the top part of Fig. 1, we show RC (right chiral) and LC (left chiral) fermions at almost the speed of light, where 1) the spin virtually aligns with the momentum (see Vol. 1, pgs. 95-96) and 2) helicity and chirality are essentially the same, i.e., the RC particle has RH (right hand helicity) and the LC particle has LH (left hand helicity).

In the lower part, LHS of the figure, we show RC and LC particles at speed much less than light, so spin does not align with momentum. The chirality of the particles remains the same, but the helicity changes (and the particles are no longer in helicity eigenstates).

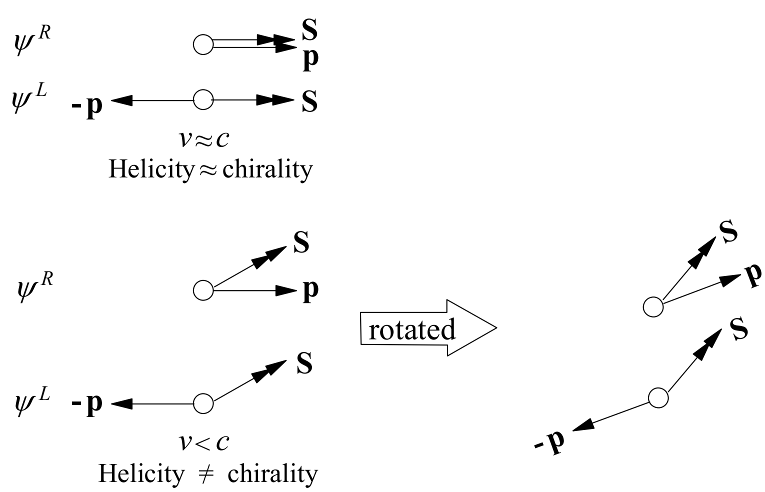


**Figure 1. Heuristic Visualization of Fermion Rotation Transformation**

In the lower part, RHS of the figure, we have rotated our reference frame. Note that both the RC and LC particles are affected in the same way, as we noted at the end of Sect. 3. The relationship between the spin and the momentum stays the same for both particles.

### 5.3 Boost Visualization

In Fig. 2, we show the same particles on the LHS as in the lower part LHS of Fig. 1. But note that when we boost both particles in the same direction, the momenta changes in opposite ways (one gets greater, the other lesser). Further, the spins change in opposite ways, as well. For an increase in momentum, spin gets closer to aligning with the direction of momentum. For a decrease in momentum, spin gets further from such alignment.

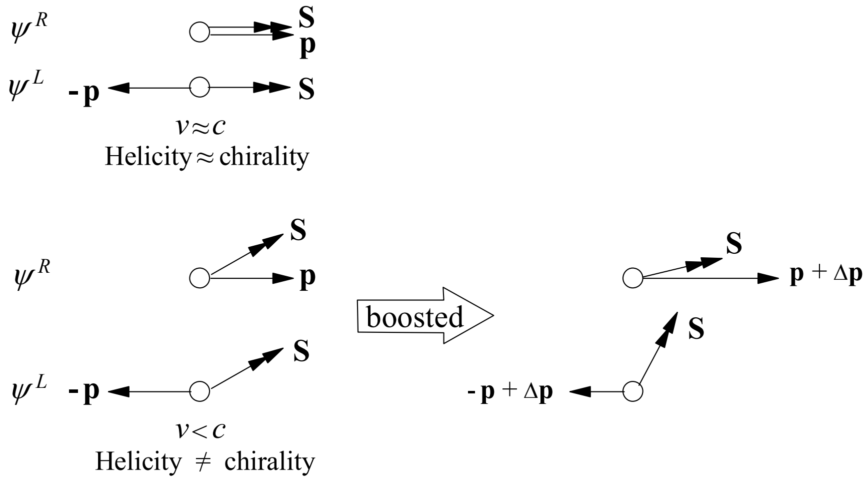


**Figure 2. Heuristic Visualization of Fermion Boost Transformation**

So, a boost transformation of an RC particle has the opposite effect of a boost on an LC particle, as we noted in Sect. 4.

## 6 Summary

Note that we have not derived the transformation (5-64). That is done in the references of the footnote of Vol. 1, pg. 171, and the derivations are long and complex. What we have done here is justify (5-64) to some degree in order to gain some intuitive level of comfort with the relation.

## Appendix

The equivalent of (5-65) for $v^k$ a substantial fraction of the speed of light is more complicated. We start by assuming our coordinate system $x^3$ is aligned with the direction of the boost **v**, and defining a parameter $\phi$ via

$$cosh\,\phi = \frac{1}{\sqrt{1-v^2}} \qquad sinh\,\phi = \frac{v}{\sqrt{1-v^2}} \qquad tanh\,\phi = v\,. \tag{16}$$

Then, $Q^k$, which we state without proof (see earlier cited references), is

$$Q^k = (0,0,\phi)\,, \tag{17}$$

and (5-64) is

$$D = e^{-i\left(L^k \Theta^k + M^k Q^k\right)} = e^{-i\left(L^k \Theta^k + M^3 Q^3\right)} = e^{-i\left(L^k \Theta^k + M^3 \phi\right)}\,. \tag{18}$$

This can be generalized to

$$D = e^{-i\left(L^k \Theta^k + M^k \phi^k\right)}\,, \tag{19}$$

where there are three $\phi^k$ for the general case of three components $v^k$.

In the limit of small $v$, from (16),

$$sinh\,\phi = \frac{v}{\sqrt{1-v^2}} \xrightarrow{v \ll 1} sinh\,\phi \approx \phi \approx v\,, \tag{20}$$

And (18) becomes

$$D = e^{-i\left(L^k \Theta^k + M^3 v\right)} = e^{-i\left(L^k \Theta^k\right)} e^{-\frac{1}{2}\gamma^0 \gamma^3 v^3}\,, \tag{21}$$

which we can generalize to (5-64) with (5-65) and (12) as

$$D = e^{-i\left(L^k \Theta^k\right)} e^{-\frac{1}{2}\gamma^0 \gamma^k v^k} \qquad v^k \ll 1\,. \tag{22}$$